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Interpolation on quadric surfaces with rational quadratic spline curves ☆

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Abstract

Given a sequence of points $\{X_i\}_{i=1}^n$ on a regular quadric S: $X^{\mathrm{T}}AX = 0 \subset \mathbb{E}^d$, $d \geqslant 3$, we study the problem of constructing a G^1 rational quadratic spline curve lying on S that interpolates $\{X_i\}_{i=1}^n$. It is shown that a necessary condition for the existence of a nontrivial interpolant is $(X_1^{\mathrm{T}}AX_2)(X_i^{\mathrm{T}}AX_{i+1}) > 0$, $i = 1, 2, \ldots, n-1$. Also considered is a Hermite interpolation problem on the quadric S: a biarc consisting of two conic arcs on S joined with G^1 continuity is used to interpolate two points on S and two associated tangent directions, a method similar to the biarc scheme in the plane (Bolton, 1975) or space (Sharrock, 1987). A necessary and sufficient condition is obtained on the existence of a biarc whose two arcs are not major elliptic arcs. In addition, it is shown that this condition is always fulfilled on a sphere for generic interpolation data.

1. Introduction

1.1. Problems

Given a sequence of points $\{X_i\}_{i=1}^n$ on a quadric $S \subset \mathbb{E}^d$, $d \geqslant 3$, we consider constructing a G^1 curve to interpolate $\{X_i\}_{i=1}^n$ such that the constructed curve lies on S. We will use the rational quadratic spline curve to solve the above problem. Clearly, the rational quadratic spline is the simplest curve possible for this problem.

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There has been much research in the literature on rational quadratic spline curves, or conic spline curves. Shape design using conic arcs is discussed in (Bookstein, 1979; Pavlidis, 1983; Pratt, 1985; Lee, 1987; Farin, 1989). Biarcs consisting of two conic arcs have also been studied. Circular biarcs in the plane are studied in (Bézier, 1972; Bolton, 1975; Sabin, 1976). Circular biarcs in 3D space are considered in (Sharrock, 1987; Rossignac and Requicha, 1987; Wang and Joe, 1992). Curve design on a sphere has been discussed by a number of researchers, e.g., (Shoemake, 1985; Pletinckx, 1989; Wang and Joe, 1993; Kim et al., 1995), for orientation interpolation in computer animation. In particular, rational curves on a sphere as well as on general quadrics are studied in (Hoschek and Seeman, 1992; Dietz et al., 1993, 1995).

The first problem we discuss is to construct a smooth rational quadratic spline curve on a quadric with a single conic arc between two consecutive data points. We show how to construct such a spline curve, and prove that if a solution exists, then all the line segments $\overline{X_iX_{i+1}}$, $i = 1, \dots, n-1$, are on the same side of S and the curve has $d-2$ degrees of freedom. The spline curve thus constructed does not have local control. The second problem we discuss is to use a *biarc* consisting of two conic arcs joined with G^1 continuity on a quadric S to interpolate two points X_0 and X_1 and tangent directions at X_0 and X_1, respectively.

The remainder of the paper is organized as follows. In the rest of this section relevant preliminaries are reviewed. Sections 2 and 3 deal with the two problems mentioned above, respectively. Section 3 also describes an algorithm which uses the biarc interpolant to interpolate a sequence of points on a quadric. Section 4 contains concluding remarks.

1.2. Preliminaries

A point in $\mathbb{E}^d$ is represented by homogeneous coordinates $X = (x_1, \dots, x_{d+1})^{\mathrm{T}}$, where the x_i are reals and at least one $x_i \neq 0$. If $x_{d+1} = 0$, X is a point at infinity with respect to $\mathbb{E}^d$. The point represented by homogeneous coordinates X is also denoted by $\langle X \rangle$.

A finite point $X = (x_1, \dots, x_{d+1})^{\mathrm{T}} \in \mathbb{E}^d$ is in *normalized homogeneous form* if $x_{d+1} = 1$. Tangent directions are represented by points at infinity. If T_0 is a point at infinity, then $-T_0$ stands for the opposite direction of T_0, though T_0 and $-T_0$ represent the same point at infinity.

A quadric $S \subset \mathbb{E}^d$ is represented by $X^{\mathrm{T}}AX = 0$, where A is a $(d+1) \times (d+1)$ real symmetric matrix. We will consider only the real *regular* quadric S, i.e., S has no singular points in real projective space. The condition for $X^{\mathrm{T}}AX = 0$ to be a real regular quadric is that A is indefinite and nonsingular. A regular quadric is irreducible, i.e., it is not composed of hyperplanes (Semple and Kneebone, 1952).

For a regular quadric S, the tangent hyperplane of S at a point $X_0 \in S$ is $X_0^{\mathrm{T}}AX = 0$. Like on a quadric surface in $\mathbb{E}^3$, if a straight line is contained entirely in S in $\mathbb{E}^d$, it is called a *generating line* of S. It is easily verified that two distinct points X_0 and X_1 on S are on the same generating line of S if and only if $X_0^{\mathrm{T}}AX_1 = 0$.

A conic that is composed of straight lines is said to be *degenerate*, otherwise *nondegenerate*. A *conic arc* refers to a G^1 continuous and finite piece of conic section, including a line segment. A *nondegenerate conic arc* refers to an arc on a nondegenerate conic; therefore there is a unique 2D plane containing a nondegenerate conic arc. A conic arc can be represented in the following *standard Bézier form* (Patterson, 1986)

$$P(u) = P_0 B_{0,2}(u) + wP_1 B_{1,2}(u) + P_2 B_{2,2}(u), \quad u \in [0, 1]. \tag{1}$$

Here P_0 and P_2 are in normalized homogeneous form. If P_1 is fixed, two weights w with opposite signs give rise to two complementary arcs of the same conic (Lee, 1987); both arcs are continuous if and only if the conic is an ellipse.

A curve segment (1) is continuous if $wx_{d+1} \geqslant 0$, where x_{d+1} is the last component of P_1. When $w = 0$ the curve becomes the line segment $\overline{P_0P_2}$; when $x_{d+1} = 0$ and $w \neq 0$ the curve is half an ellipse. In the following we will mainly consider the case $w \neq 0$, as it will be shown later on that straight line segments do not appear in a general conic spline curve on a regular quadric.

Definition 1.1. Let x_{d+1} be the last component of P_1 in (1). A weight $w \neq 0$ is a proper weight if $wx_{d+1} > 0$ or $w > 0$ and $x_{d+1} = 0$; it is a *complementary weight* if $wx_{d+1} < 0$ or $w < 0$ and $x_{d+1} = 0$.

Let the control polygons of two Bézier segments be $X_0Y_0X_1$ and $X_1Y_1X_2$ respectively. Then we have the following result, whose trivial proof is omitted.

Lemma 1.2. *Suppose Y_0, X_1 and Y_1 are collinear. When the joint point X_1 is between Y_0 and Y_1, the two Bézier curves join smoothly if and only if they both take the proper weights or the complementary weights simultaneously. When X_1 is not between Y_0 and Y_1, the two Bézier curves join smoothly if and only if one of two curves takes the proper weight and the other takes the complementary weight.*

2. Point interpolation on a quadric

2.1. Local representation

Let $\{X_i\}_{i=1}^n$, $n \geqslant 3$, be a point sequence in normalized homogeneous form on a quadric S: $X^{\mathrm{T}}AX = 0 \subset \mathbb{E}^d$, $d \geqslant 3$. Assume that $\{X_i\}_{i=1}^n$ are on the same real component of S. Our goal is to construct a G^1 rational quadratic spline curve on S to interpolate $\{X_i\}_{i=1}^n$. First we consider the existence and properties of a single rational quadratic Bézier curve on S interpolating two consecutive points X_i and X_{i+1}, with i fixed.

Let the tangent hyperplane of S at X_i and X_{i+1} be Q_i: $X_i^{\mathrm{T}}AX = 0$ and Q_{i+1}: $X_{i+1}^{\mathrm{T}}AX = 0$, respectively. Let L_i be the intersection of Q_i and Q_{i+1}, which is a $(d-2)$-dimensional affine manifold. Let C_i: $P_i(u)$ be a rational quadratic Bézier curve on S interpolating X_i and X_{i+1}. Let X_i, Y_i and X_{i+1} be the control points of $P_i(u)$ in Bézier form. Since C_i is on S, Y_i is necessarily on L_i, i.e., $X_i^{\mathrm{T}}AY_i = 0$ and $X_{i+1}^{\mathrm{T}}AY_i = 0$; for otherwise the straight line Y_iX_i or Y_iX_{i+1} would not be tangent to S, contradictory to $C_i \subset S$.

Let the standard Bézier representation of $P_i(u)$ be

$$P_i(u) = X_i B_{0,2}(u) + wY_i B_{1,2}(u) + X_{i+1} B_{2,2}(u), \quad u \in [0,1]. \tag{2}$$

The weight w must satisfy $P_i(u)^{\mathrm{T}} A P_i(u) = 0$ for all $u \in [0,1]$ since $C_i \subset S$. Substituting (2) in $P_i(u)^{\mathrm{T}} A P_i(u) = 0$, noting that $X_i^{\mathrm{T}} A X_i = X_{i+1}^{\mathrm{T}} A X_{i+1} = X_i^{\mathrm{T}} A Y_i = X_{i+1}^{\mathrm{T}} A Y_i = 0$, we obtain

$$2X_i^{\mathrm{T}} A X_{i+1} B_{0,2}(u) B_{2,2}(u) + w^2 Y_i^{\mathrm{T}} A Y_i B_{1,2}^2(u) = 0,$$

or, when $Y_i^{\mathrm{T}} A Y_i \neq 0$, as $B_{1,2}^2(u) = 4B_{0,2}(u) B_{2,2}(u)$, there is

$$w^2 = -\frac{X_i^{\mathrm{T}} A X_{i+1}}{2Y_i^{\mathrm{T}} A Y_i}. \tag{3}$$

When the right hand side of (3) is nonnegative, a real value of w can be solved for from (3). Now we shall find the condition on Y_i for the right-hand side of (3) to be nonnegative.

Lemma 2.1. *Let X_i and X_{i+1} be distinct points on the same generating line of the quadric S. Then the line segment $\overline{X_i X_{i+1}}$ is the only conic arc on S interpolating X_i and X_{i+1}.*

Proof. Suppose there is another conic C_i passing through X_i and X_{i+1}, which is necessarily nondegenerate. Then the unique plane containing C_i intersects the quadric S in the conic C_i plus the line $X_i X_{i+1}$, contradicting that any plane section of a quadric is a conic if the plane is not contained in the quadric. □

Lemma 2.2. *On a regular quadric S a straight line segment and a nondegenerate conic arc cannot meet with G^1 continuity.*

Proof. Suppose a nondegenerate conic arc C and a straight line segment ℓ on S join with common tangent T. Let P_C be the plane determined by C. Then P_C contains T, and therefore contains ℓ. So the plane P_C intersects the quadric S in a cubic curve consisting of the conic containing C plus the straight line containing ℓ. This is a contradiction. □

Because of Lemma 2.1 and Lemma 2.2, the case where two consecutive points X_i and X_{i+1} are on the same generating line of S is not of interest to us, and will therefore be excluded.

The following theorem provides a geometric condition on the existence of a local interpolating rational quadratic curve and also a way to construct it.

Theorem 2.3. *Let S: $X^{\mathrm{T}} A X = 0 \subset \mathbb{E}^d$ be a regular quadric. Let X_i and $X_{i+1} \in S$ be distinct points on the same component but not on the same generating line of S. Then, X_i, Y_i and X_{i+1} are the control points of a rational quadratic Bézier curve on S interpolating X_i and X_{i+1} if and only if $X_i^{\mathrm{T}} A Y_i = X_{i+1}^{\mathrm{T}} A Y_i = 0$ and $(X_i^{\mathrm{T}} A X_{i+1})(Y_i^{\mathrm{T}} A Y_i) < 0$, or geometrically,* (i) *$Y_i \in L_i$ and* (ii) *the point Y_i and the line segment $\overline{X_i X_{i+1}}$ are on the opposite sides of S.*

Proof. Let X_i and X_{i+1} be in normalized homogeneous form. Then $X_i + X_{i+1}$ is a point on the line segment $\overline{X_iX_{i+1}}$. The segment $\overline{X_iX_{i+1}}$ is entirely on the same side of S as the point $\langle X_i + X_{i+1}\rangle$, since X_i and X_{i+1} are the only intersections of the straight line X_iX_{i+1} with S.

When X_i, Y_i and X_{i+1} are the control points of a Bézier curve $P_i(u)$ of the form (2) on S, $Y_i \in L_i$, i.e., $X_i^{\mathrm{T}}AY_i = X_{i+1}^{\mathrm{T}}AY_i = 0$. From the existence of $P_i(u)$ connecting X_i and X_{i+1}, by (3) we have $-X_i^{\mathrm{T}}AX_{i+1}/(2Y_i^{\mathrm{T}}AY_i) = w^2 \geqslant 0$. Since X_i and X_{i+1} are not on the same generating line of S, $X_i^{\mathrm{T}}AX_{i+1} \neq 0$. Therefore $(X_i^{\mathrm{T}}AX_{i+1})(Y_i^{\mathrm{T}}AY_i) < 0$. As $(X_i + X_{i+1})^{\mathrm{T}}A(X_i + X_{i+1}) = 2X_i^{\mathrm{T}}AX_{i+1}$, we have

$$\left[(X_i + X_{i+1})^{\mathrm{T}}A(X_i + X_{i+1})\right](Y_i^{\mathrm{T}}AY_i) < 0.$$

Hence Y_i and $\overline{X_iX_{i+1}}$ are on the opposite sides of S.

Now suppose that (i) $X_i^{\mathrm{T}}AY_i = X_{i+1}^{\mathrm{T}}AY_i = 0$ and (ii) $(Y_i^{\mathrm{T}}AY_i)(X_i^{\mathrm{T}}AX_{i+1}) < 0$. Since $Y_i \in L_i$, by (i), we can construct a Bézier curve on S of the form (2), with the weight w determined by (3). By (ii), we have

$$\frac{-X_i^{\mathrm{T}}AX_{i+1}}{2Y_i^{\mathrm{T}}AY_i} > 0.$$

Therefore a proper w can be solved for from (3), i.e., X_i, Y_i and X_{i+1} are the control points of a Bézier curve on S interpolating X_i and X_{i+1}. □

It is evident that when X_i and X_{i+1} are distinct points on the sphere $S^{d-1} \subset \mathbb{E}^d$ or any surface that is affinely equivalent to S^{d-1}, $(X_i^{\mathrm{T}}AX_{i+1})(Y_i^{\mathrm{T}}AY_i) < 0$ holds for any $Y_i \in L_i$. Therefore we have

Lemma 2.4. *Let X_i and X_{i+1} be two distinct points on a quadric S that is affinely equivalent to the sphere $S^{d-1} \subset \mathbb{E}^d$. Then for any $Y_i \in L_i$, the three points X_i, Y_i and X_{i+1} are the control points of two rational quadratic Bézier curves interpolating X_i and X_{i+1} on S^{d-1}, one with the proper weight and the other with the complementary weight.*

For a general regular quadric we have only a weaker result. From Theorem 2.3 it is seen that $Y_i \in L_i$ gives an interpolating Bézier curve (2) if and only if the right-hand side of (3) is nonnegative.

Lemma 2.5. *Let X_i and X_{i+1} be distinct points on the same component but not on the same generating line of a regular quadric S: $X^{\mathrm{T}}AX = 0 \subset \mathbb{E}^d$, $d \geqslant 3$. Then there exists a conic arc on S that connects X_i and X_{i+1}.*

Proof. See the Appendix. □

Lemma 2.5 cannot be made as strong as Lemma 2.4 because on a hyperboloid of one sheet S in $\mathbb{E}^3$ it is easy to give two points X_i and $X_{i+1} \in S$ and a $Y_i \in L_i$ such that Y_i and $\overline{X_iX_{i+1}}$ are on the same side of S. See Fig. 1.

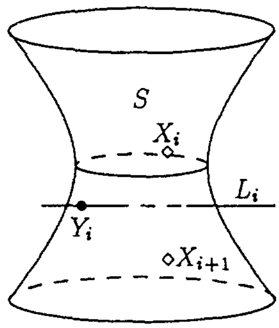

Fig. 1. The points X_i and X_{i+1} are on the front side of hyperboloid S and the solid part of L_i is outside S. The point $Y_i \in L_i$ is on the same side of S as the segment $\overline{X_i X_{i+1}}$.

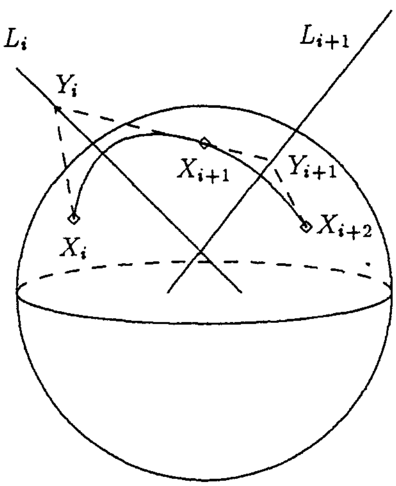

Fig. 2. The control point $Y_{i+1} \in L_{i+1}$ is the projection of $Y_i \in L_i$ through X_{i+1}. The points X_i, X_{i+1}, and X_{i+2} are on the front side of the sphere and the intersection of L_i and L_{i+1} is in front of the sphere.

2.2. *Construction of interpolating spline curves*

Given $\{X_i\}_{i=1}^n$, $n \geqslant 3$, on S: $X^{\mathrm{T}}AX = 0 \subset \mathbb{E}^d$, we now consider constructing a G^1 rational quadratic spline curve on S interpolating $\{X_i\}_{i=1}^n$. Let X_i, Y_i and X_{i+1} be the control points of the local curve segment C_i in the standard Bézier representation (2). We need to determine all the Y_i so that C_i and C_{i+1} join with G^1 continuity, $i = 1, \ldots, n-1$. To have a well defined problem we assume that any two consecutive points are distinct and $\{X_i\}_{i=1}^n$ are on the same component of S. As explained earlier, we assume that no two consecutive points are on the same generating line of S.

Now given $\{X_i\}_{i=1}^n$, by Lemma 2.5, we can first choose $Y_1 \in L_1$ such that (3) is valid. Let us now find Y_{i+1} with Y_i being known. Since $\overline{Y_i X_{i+1}}$ and $\overline{Y_{i+1} X_{i+1}}$ are the tangents to C_i and C_{i+1} at their joint point X_{i+1}, respectively, in order for C_i and C_{i+1} to have common tangent at X_{i+1}, the point Y_{i+1} must be the projection of $Y_i \in L_i$ through X_{i+1} into L_{i+1}. See Fig. 2 for illustration. So Y_{i+1} depends projectively on Y_1. The following lemma gives the expression of this dependence.

Lemma 2.6. *Let $M_i = \prod_{j=1}^{i} R_j$, with $R_1 = I$, the identity matrix, and*

$$R_j = X_j X_{j+1}^{\mathrm{T}} A - (X_j^{\mathrm{T}} A X_{j+1}) I, \quad j = 2, \dots, n-1.$$

If the interpolating quadratic spline curve exists, then $Y_i = M_i Y_1$, $i = 1, \dots, n-1$.

Proof. Because Y_i, X_{i+1} and Y_{i+1} are collinear, we have

$$Y_{i+1} = aX_{i+1} + bY_i$$

for some constants a and b. Premultiplying $X_{i+2}^{\mathrm{T}} A$ to both sides, since $X_{i+2}^{\mathrm{T}} A Y_{i+1} = 0$, we obtain

$$0 = a(X_{i+2}^{\mathrm{T}} A X_{i+1}) + b(X_{i+2}^{\mathrm{T}} A Y_i).$$

So omitting a nonzero multiplicative constant, we have

$$\begin{aligned} Y_{i+1} &= (X_{i+2}^{\mathrm{T}} A Y_i) X_{i+1} - (X_{i+2}^{\mathrm{T}} A X_{i+1}) Y_i \\ &= \left[X_{i+1} X_{i+2}^{\mathrm{T}} A - (X_{i+1}^{\mathrm{T}} A X_{i+2}) I\right] Y_i. \end{aligned} \tag{4}$$

Let $R_j = X_j X_{j+1}^{\mathrm{T}} A - (X_j^{\mathrm{T}} A X_{j+1}) I$, $j = 2, \dots, n-1$. Then the lemma follows. □

The next theorem gives a necessary condition on the existence of a rational quadratic spline curve interpolating $\{X_i\}_{i=1}^{n}$.

Theorem 2.7. *Let a sequence of points $\{X_i\}_{i=1}^{n}$ be given on the same component of a regular quadric S: $X^{\mathrm{T}} A X = 0 \subset \mathbb{E}^d$, $d \geqslant 3$. Assume that no two consecutive points X_i and X_{i+1} are on the same generating line of S. A necessary condition for the existence of a G^1 rational quadratic spline curve on S interpolating $\{X_i\}_{i=1}^{n}$ is that $(X_1^{\mathrm{T}} A X_2)(X_i^{\mathrm{T}} A X_{i+1}) > 0$, $i = 1, \dots, n-1$, i.e., all the line segments $\overline{X_i X_{i+1}}$ are on the same side of S.*

The next lemma is needed in the proof of the above theorem.

Lemma 2.8. *Let $\{X_i\}_{i=1}^{n}$ be given as in Theorem 2.7. Let $Y_1 \in L_1$ and $Y_i = M_i Y_1$, $i = 2, \dots, n-1$, as defined in Lemma 2.6. If $Y_1^{\mathrm{T}} A Y_1 \neq 0$, then $(Y_1^{\mathrm{T}} A Y_1)(Y_i^{\mathrm{T}} A Y_i) > 0$, $i = 1, 2, \dots, n-1$, i.e., all the Y_i are on the same side of S.*

Proof. As obtained in the proof of Lemma 2.6,

$$Y_{i+1} = (X_{i+2}^{\mathrm{T}} A Y_i) X_{i+1} - (X_{i+1}^{\mathrm{T}} A X_{i+2}) Y_i, \quad i = 1, \dots, n-2.$$

Since $X_{i+1}^{\mathrm{T}} A X_{i+1} = X_{i+1}^{\mathrm{T}} A Y_i = 0$, it follows from the above expression that

$$Y_{i+1}^{\mathrm{T}} A Y_{i+1} = (X_{i+1}^{\mathrm{T}} A X_{i+2})^2 (Y_i^{\mathrm{T}} A Y_i).$$

Since X_{i+1} and X_{i+2} are not on the same generating line of S, $(X_{i+1}^{\mathrm{T}}AX_{i+2})^2 > 0$. Hence the lemma follows. □

Proof of Theorem 2.7. By Lemma 2.8, all the points Y_i are on the same side of S. By Theorem 2.3, for any i, the line segment $\overline{X_iX_{i+1}}$ and the point Y_i are on the opposite sides of S. Hence all the line segments $\overline{X_iX_{i+1}}$ are on the same side of S, i.e., $(X_1^{\mathrm{T}}AX_2)(X_i^{\mathrm{T}}AX_{i+1}) > 0$, $i = 1, \ldots, n-1$. □

The condition given in Theorem 2.7 is in general not sufficient. However, on a sphere we have

Theorem 2.9. *Given a point sequence $\{X_i\}_{i=1}^n$ on $S \subset \mathbb{E}^d$ which is affinely equivalent to the sphere S^{d-1}, for any point $Y_1 \in L_1$, there exists a G^1 rational quadratic spline curve on S interpolating $\{X_i\}_{i=1}^n$, with the initial control point being Y_1.*

Proof. Let $Y_1 \in L_1$ and $Y_i = M_iY_1$, $i = 2, \ldots, n-1$, be given as in Lemma 2.6. By Theorem 2.3 and Lemma 2.4, for any $Y_1 \in L_1$ we have $-(X_1^{\mathrm{T}}AX_2)/(2Y_1^{\mathrm{T}}AY_1) > 0$ since $X_1^{\mathrm{T}}AX_2 \neq 0$. By Lemma 2.8, $(Y_1^{\mathrm{T}}AY_1)(Y_i^{\mathrm{T}}AY_i) > 0$, $i = 1, 2, \ldots, n-1$. Since S is affinely equivalent to a sphere and $X_i \neq X_{i+1}$, it is easy to verify that $(X_1^{\mathrm{T}}AX_2)(X_i^{\mathrm{T}}AX_{i+1}) > 0$, i.e., all the line segments $\overline{X_iX_{i+1}}$ are on the same side of S. Therefore $-(X_i^{\mathrm{T}}AX_{i+1})/(2Y_i^{\mathrm{T}}AY_i) > 0$, $i = 1, 2, \ldots, n-1$. Hence two real weights can be obtained from (3) for each i, and both of these weights yield a continuous and smooth Bézier curve segment since the underlying conic is an ellipse. So the required interpolating spline curve is given by applying Lemma 1.2 to choose the appropriate weights successively to ensure G^1 continuity between all adjacent conic arcs. □

The condition in Theorem 2.7 imposes a substantial restriction on a general quadric. For example, on a hyperboloid of one sheet S in $\mathbb{E}^3$, it is easy to come up with a point sequence $\{X_i\}_{i=1}^n$ such that not all the line segments $\overline{X_iX_{i+1}}$ are on the same side of S. See Fig. 3. Hence by Theorem 2.7 it is impossible in this case to construct a G^1 rational quadratic spline curve on S to interpolate $\{X_i\}_{i=1}^n$.

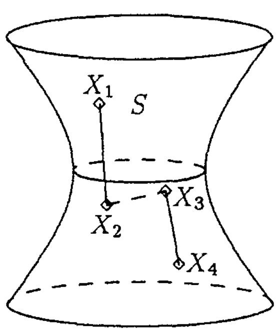

Fig. 3. Not all the line segments connecting consecutive data points are on the same side of hyperboloid S.

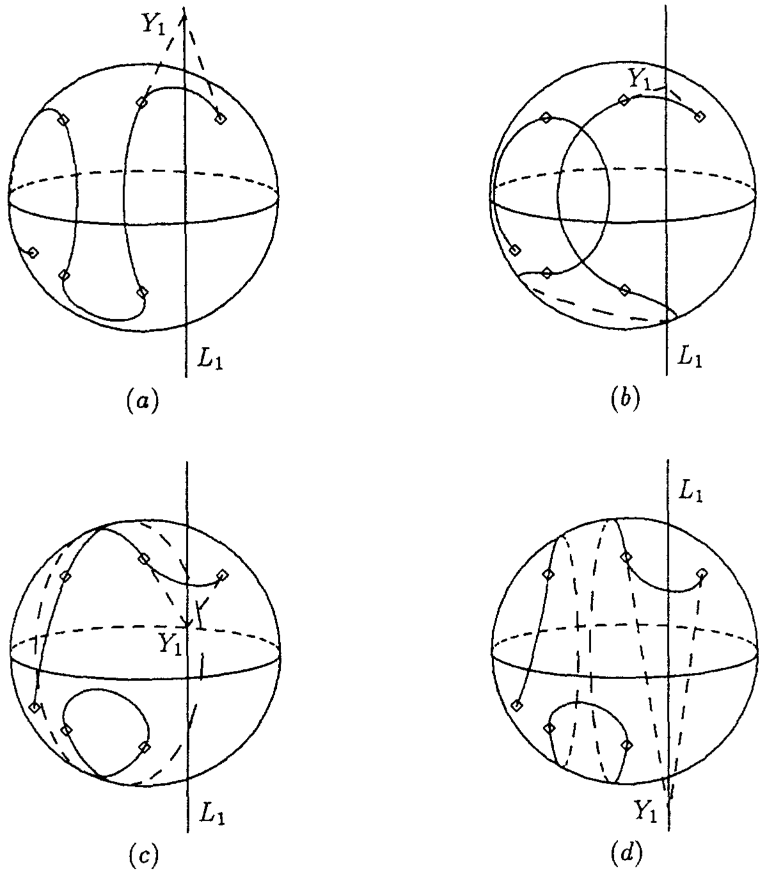

Fig. 4. Four spline curves interpolating the same set of data points are given by different $Y_1 \in L_1$.

Fig. 4 illustrates the application of the above method to interpolating six data points on a sphere in $\mathbb{E}^3$ by choosing different points $Y_1 \in L_1$. Still we do not know how to choose the best Y_1 or if there is always an acceptable choice of Y_1 for all possible data. This is mainly because Y_1 has global influence over the whole curve. Later on we will see that biarc interpolants provide a better solution with local control.

2.3. Closed interpolating spline curves

Given points $\{X_i\}_{i=1}^{n+2}$, $n \geqslant 3$, on S: $X^{\mathrm{T}}AX = 0$ with $X_{n+1} = X_1$ and $X_{n+2} = X_2$, we now consider constructing a G^1 rational quadratic spline curve on S interpolating $\{X_i\}_{i=1}^{n+2}$. Clearly, such a spline curve induces a closed G^1 curve interpolating $\{X_i\}_{i=1}^{n}$, by just removing its last curve segment. In order for this problem to have a solution, it is necessary that there exist $Y_1 \in L_1$ such that $M_{n+1}Y_1 = \rho Y_1$ for some $\rho \neq 0$, where M_{n+1} is defined in Lemma 2.6.

From its definition in Lemma 2.6, $M_i = \prod_{j=1}^{i} R_j$, where R_j induces a projection from L_{j-1} to L_j. Therefore M_i, when restricted to L_1, is a projective transformation from L_1 to L_i; in particular, M_{n+1} induces a projective transformation on L_1. Thus the following is evident.

Lemma 2.10. *There exists a closed rational quadratic spline curve interpolating $\{X_i\}_{i=1}^{n}$ if and only if there exists a G^1 rational quadratic spline curve interpolating $\{X_i\}_{i=1}^{n+2}$, $X_{n+1} = X_1$ and $X_{n+2} = X_2$, with the initial control point $Y_1 \in L_1$ such that Y_1 is a real fixed point of M_{n+1}.*

The condition on the existence of a real fixed point of M_{n+1} in L_1 is still unknown in general. Now let us consider the cases of $d = 3$ and $d = 4$. When $d = 3$, L_1 is a straight line in $\mathbb{E}^3$, and M_{n+1} induces a homography $H(L_1)$ on L_1. A *homography* on a straight line is a rational linear transformation on it. A *united point* of a homography is one of its fixed points on the straight line. By the theory of homography on a straight line (Semple and Kneebone, 1952), $H(L_1)$ has either two distinct real united points, or a double real united point, or a pair of conjugate complex united points. So M_{n+1} does not always have real fixed points on L_1.

When $d = 4$, L_1 is a 2-dimensional plane in $\mathbb{E}^4$.

Lemma 2.11. *When $d = 4$, M_{n+1} always has a real fixed point on the plane L_1.*

Proof. First establish a projective frame of reference F in L_1. Then the transformation induced by M_{n+1} on L_1 can be represented by a nonsingular 3×3 real matrix M with reference to F. Such a matrix has a nonzero real eigenvalue and an associated real eigenvector, and this eigenvector gives a real fixed point $Y_1 \in L_1$ of M_{n+1}. □

Thus, in particular, for the closed interpolation problem on $S^3 \subset \mathbb{E}^4$ we have

Theorem 2.12. *Let $\{X_i\}_{i=1}^{n}$ be a point sequence on the sphere $S^3 \subset \mathbb{E}^4$. There exists a G^1 closed rational quadratic spline curve interpolating $\{X_i\}_{i=1}^{n}$ on S^3.*

Proof. By Lemma 2.11, $Y_1 \in L_1$ can be chosen to be a real fixed point of M_{n+1}. By Theorem 2.9 there is a G^1 rational quadratic spline curve interpolating $\{X_i\}_{i=1}^{n+2}$ with the initial point being Y_1, where $X_{n+1} = X_1$, $X_{n+2} = X_2$. So by Lemma 2.10 there is a closed rational quadratic spline curve interpolating $\{X_i\}_{i=1}^{n}$. □

3. Biarc interpolation on a quadric

In this section we consider the following biarc interpolation problem on a quadric S. Let X_0 and X_1 be two distinct points in normalized homogeneous form on S: $X^{\mathrm{T}}AX = 0 \subset \mathbb{E}^d$, $d \geqslant 3$. Assume that X_0 and X_1 are on the same component but not on the same generating line of S. Let T_0 and T_1 be the tangent directions, represented as points at infinity, to be interpolated at X_0 and X_1, respectively. The problem is to find a biarc on

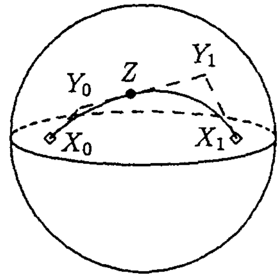

Fig. 5. A spherical biarc with control points.

S interpolating the data $D = \{X_0, X_1, T_0, T_1\}$. Naturally we assume that T_0 and T_1 are also tangent to S. Thus $X_0^T A T_0 = 0$ and $X_1^T A T_1 = 0$.

A *biarc* on S is a curve consisting of two rational quadratic Bézier curves (or conic arcs) joined with G^1 continuity. We mainly consider a special kind of biarcs consisting of rational quadratic Bézier curves with proper weights, which will be called the *biarcs with proper weights*. This is because the complementary arc of a conic arc with proper weight is not continuous, unless the underlying conic is elliptic.

3.1. Biarcs with proper weights

Let C_0 and C_1 be the two conic arcs of a biarc with proper weights on S with standard Bézier representations $P_0(u)$ and $P_1(v)$, respectively. Let the control points of $P_0(u)$ and $P_1(v)$ be X_0, Y_0, Z and Z, Y_1, X_1, where Z is the joint of the two arcs (see Fig. 5). Denote the three tangent hyperplanes of S at X_0, X_1 and Z by, respectively, Q_0: $X_0^T A X = 0$, Q_1: $X_1^T A X = 0$ and Q: $Z^T A X = 0$. Then Y_0 must be on the $(d-2)$-dimensional affine manifold $L_0 \equiv Q_0 \cap Q$ defined by $X_0^T A X = Z^T A X = 0$. Similarly $Y_1 \in L_1 \equiv Q \cap Q_1$, where L_1 is defined by $Z^T A X = X_1^T A X = 0$. The points Y_0, Z and Y_1 are assumed to be collinear, in order for C_0 and C_1 to join smoothly at Z. Let

$$Y_0 = X_0 + k_0 T_0 \quad \text{and} \quad Y_1 = X_1 - k_1 T_1, \tag{5}$$

where $k_0, k_1 > 0$. The assumption that $k_0, k_1 > 0$ follows from that only biarcs with proper weights are considered. Consequently, by Lemma 1.2, the joint Z is between Y_0 and Y_1. Now we assume that $\langle Y_0 \rangle \neq \langle Y_1 \rangle$, so the straight line $Y_0 Y_1$ is uniquely defined; the case of $\langle Y_0 \rangle = \langle Y_1 \rangle$ will be discussed later on.

Since $X_0^T A X_0 = X_0^T A T_0 = X_1^T A X_1 = X_1^T A T_1 = 0$, from (5) we have

$$Y_0^T A Y_0 = k_0^2 T_0^T A T_0 \quad \text{and} \quad Y_1^T A Y_1 = k_1^2 T_1^T A T_1. \tag{6}$$

Because a solution to the above biarc interpolation problem can be regarded as a solution to the point interpolation problem discussed in the last section for the data points X_0, Z and X_1, we obtain the following necessary condition.

Lemma 3.1. *A necessary condition for the biarc interpolation problem to be solvable is*

$$(T_0^T A T_0)(T_1^T A T_1) > 0.$$

Proof. When the problem is solvable, by Lemma 2.8, $(Y_0^{\mathrm{T}}AY_0)(Y_1^{\mathrm{T}}AY_1) > 0$. By (6), since $k_0^2k_1^2 > 0$, we obtain $(T_0^{\mathrm{T}}AT_0)(T_1^{\mathrm{T}}AT_1) > 0$. □

Because of Lemma 3.1, without loss of generality, we can normalize T_0 and T_1, replacing A by $-A$ if necessary, so that $T_0^{\mathrm{T}}AT_0 = T_1^{\mathrm{T}}AT_1 = 1$. So we will assume that T_0 and T_1 are given satisfying $T_0^{\mathrm{T}}AT_0 = T_1^{\mathrm{T}}AT_1 = 1$. Then (6) can be written as

$$Y_0^{\mathrm{T}}AY_0 = k_0^2 \quad \text{and} \quad Y_1^{\mathrm{T}}AY_1 = k_1^2. \tag{7}$$

By the preceding observation regarding the relation between Y_0, Z and Y_1, the straight line Y_0Y_1 is well defined and Z is the tangent point of the line Y_0Y_1 to S. Thus the Joachimsthal's equation [19] obtained by substituting the parametric representation $\lambda Y_0 + \mu Y_1$ of Y_0Y_1 in $X^{\mathrm{T}}AX = 0$,

$$\lambda^2(Y_0^{\mathrm{T}}AY_0) + 2\lambda(Y_0^{\mathrm{T}}AY_1) + \mu^2(Y_1^{\mathrm{T}}AY_1) = 0,$$

has a double root. Therefore its discriminant

$$\Delta \equiv 4\left[(Y_0^{\mathrm{T}}AY_1)^2 - (Y_0^{\mathrm{T}}AY_0)(Y_1^{\mathrm{T}}AY_1)\right] = 0,$$

or, by (7),

$$(Y_0^{\mathrm{T}}AY_1)^2 - k_0^2k_1^2 = 0.$$

Then it follows that

$$Y_0^{\mathrm{T}}AY_1 - k_0k_1 = 0 \tag{8}$$

or

$$Y_0^{\mathrm{T}}AY_1 + k_0k_1 = 0. \tag{9}$$

When $\Delta = 0$, $\lambda/\mu = -(Y_0^{\mathrm{T}}AY_1)/(Y_0^{\mathrm{T}}AY_0)$. Thus, omitting a nonzero multiplicative factor, the straight line $\lambda Y_0 + \mu Y_1$ touches S at

$$Z = (Y_0^{\mathrm{T}}AY_1)Y_0 - (Y_0^{\mathrm{T}}AY_0)Y_1.$$

By (8) or (9) we obtain, respectively,

$$Z = k_0k_1Y_0 - k_0^2Y_1 \tag{10}$$

or

$$Z = -k_0k_1Y_0 - k_0^2Y_1. \tag{11}$$

Since Y_0 and Y_1 are in normalized homogeneous form and Z is required to lie between Y_0 and Y_1, we discard (10) and retain (11) as the desired expression for Z, because when $k_0, k_1 > 0$, (10) gives a point Z outside the line segment $\overline{Y_0Y_1}$; however, when $k_0 > 0$ and $k_1 > 0$, i.e., when proper weights are used, by Lemma 1.2, Z must be a point on $\overline{Y_0Y_1}$. Dividing by $-k_0$ in (11) yields

$$\begin{aligned} Z(k_0, k_1) &= k_1Y_0 + k_0Y_1 = k_1(X_0 + k_0T_0) + k_0(X_1 - k_1T_1) \\ &= k_1[X_0 + k_0(T_0 - T_1)] + k_0X_1. \end{aligned} \tag{12}$$

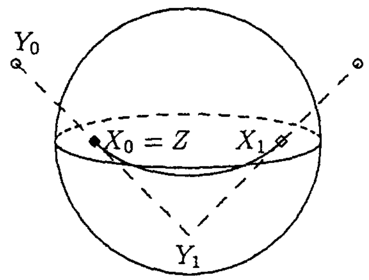

Fig. 6. An instance of singular data on S^2 is shown with one of its degenerate biarc interpolants and the control polygons. The joint point Z is marked with •, which coincides with X_0. The ends of tangents T_0 and T_1 are marked with ∘.

Substituting (5) in (9), k_0 and k_1 are found to be related by

$$X_0^T A X_1 + k_0 X_1^T A T_0 - k_1 X_0^T A T_1 + k_0 k_1 (1 - T_0^T A T_1) = 0. \tag{13}$$

In the above derivation it is assumed that $\langle Y_0 \rangle \neq \langle Y_1 \rangle$; for otherwise the straight line $Y_0 Y_1$ is not uniquely defined. It will be shown that $\langle Y_0 \rangle = \langle Y_1 \rangle$ occurs for some k_0 and k_1 satisfying (13) only when D is the data of a special kind.

Definition 3.2. The data $D = \{X_0, X_1, T_0, T_1\}$ is *singular* if $X_0 + \rho T_0 = X_1 + \rho T_1$ for some finite $\rho \neq 0$ or $T_0 = T_1$.

Let $[X, T)$ denote a half line starting at X and pointing in the direction T. Then geometrically, for singular data with $T_0 \neq T_1$, the half lines $[X_0, T_0)$ and $[X_1, T_1)$ intersect each other or the half lines $[X_0, -T_0)$ and $[X_1, -T_1)$ intersect each other. Note that if D is singular then X_0, T_0, X_1, and T_1, being treated as points in the projective space, are coplanar. An example of singular data is illustrated in Fig. 6.

Lemma 3.3. *Given data $D = \{X_0, X_1, T_0, T_1\}$ on a regular quadric S, there is $\langle Y_0 \rangle = \langle Y_1 \rangle$ for some k_0 and k_1 satisfying* (13) *if and only if D is singular.*

Proof. First consider necessity. There are two cases to consider: (i) $\langle Y_0 \rangle = \langle Y_1 \rangle$ is a finite point; (ii) $\langle Y_0 \rangle = \langle Y_1 \rangle$ is a point at infinity.

(i) In this case k_0 and k_1 are finite and $\langle Y_0 \rangle = \langle Y_1 \rangle$ implies that $Y_0 = Y_1$. Since X_0 and X_1 are distinct points, $k_0 \neq 0$ or $k_1 \neq 0$; for otherwise from $Y_0 = Y_1$ and (5), $X_0 = X_1$ would result, a contradiction. First assume $k_0 \neq 0$. By (7), $Y_0^T A Y_0 = k_0^2$. On the other hand, since $Y_0 = Y_1$, and k_0 and k_1 satisfy (9), which is equivalent to (13), $Y_0^T A Y_0 = Y_0^T A Y_1 = -k_0 k_1$. Therefore $k_0^2 = -k_0 k_1$, or $k_0 = -k_1$ since $k_0 \neq 0$. From $Y_0 = Y_1$, we obtain

$$X_0 + k_0 T_0 = X_1 - k_1 T_1 = X_1 + k_0 T_1.$$

So, by definition, D is singular. When $k_1 \neq 0$, the same conclusion follows from a similar argument.

(ii) In this case k_0 and k_1 are infinite. From $\langle Y_0 \rangle = \langle Y_1 \rangle$, we have either $T_0 = T_1$ or $T_0 = -T_1$. Eq. (13) can be rewritten as

$$\frac{X_0^T A X_1}{k_0 k_1} + \frac{X_1^T A T_0}{k_1} - \frac{X_0^T A T_1}{k_0} + 1 - T_0^T A T_1 = 0,$$

which is reduced to $1 - T_0^T A T_1 = 0$ when $k_0 = \infty$ and $k_1 = \infty$. Since $1 - T_0^T A T_1 = 0$ is satisfied by $T_0 = T_1$ but not $T_0 = -T_1$, we have $T_0 = T_1$. Hence D is singular.

Now we prove sufficiency. Suppose that D is singular. When $X_0 + \rho T_0 = X_1 + \rho T_1$ for some finite $\rho \neq 0$, it can be verified directly that $k_0 = \rho$ and $k_1 = -\rho$ satisfy (13), and $Y_0 = Y_1$ for this pair of k_0 and k_1. When $T_0 = T_1$, as above it can be shown again that $k_0 = \infty$ and $k_1 = \infty$ satisfy (13). For this pair of k_0 and k_1, we have $\langle Y_0 \rangle = \langle Y_1 \rangle = \langle T_0 \rangle = \langle T_1 \rangle$. □

Theorem 3.4. *Let $D = \{X_0, X_1, T_0, T_1\}$ be nonsingular data on a regular quadric S: $X^T A X = 0 \subset \mathbb{E}^d$ with $T_0^T A T_0 = T_1^T A T_1 = 1$. There exists a biarc with proper weights interpolating D if and only if there are solutions k_0 and k_1 of Eq.* (13) *that satisfy $k_0 > 0$, $k_1 > 0$ and*

$$X_0^T A X_1 - k_1 X_0^T A T_1 < 0, \qquad X_1^T A X_0 + k_0 X_1^T A T_0 < 0.$$

The last two conditions are equivalent to $(Y_0^T A Y_0)(X_0^T A Z) < 0$ and $(Y_1^T A Y_1)(X_1^T A Z) < 0$.

Proof. For the necessity suppose there is a biarc with proper weights interpolating D. Since the two arcs of the biarc both have proper weights, in order for T_0 and T_1 to be interpolated, we must have $k_0 > 0$ and $k_1 > 0$. By Theorem 2.3, the existence of this biarc implies that $(Y_0^T A Y_0)(X_0^T A Z) < 0$ and $(Y_1^T A Y_1)(X_1^T A Z) < 0$. Since $Z = k_1 Y_0 + k_0 Y_1$, we have

$$X_0^T A Z = k_0 X_0^T A Y_1 = k_0 (X_0^T A X_1 - k_1 X_0^T A T_1).$$

Since $k_0 > 0$ and $(Y_0^T A Y_0) = k_0^2$, from $(Y_0^T A Y_0)(X_0^T A Z) < 0$ it follows that $X_0^T A X_1 - k_1 X_0^T A T_1 < 0$. Similarly we can show $X_1^T A X_0 + k_0 X_1^T A T_0 < 0$.

To prove sufficiency, we observe that, when the conditions are satisfied, the joint $Z = k_1 Y_0 + k_0 Y_1$ is on the line segment $\overline{Y_0 Y_1}$, where $Y_0 = X_0 + k_0 T_0$ and $Y_1 = X_1 - k_1 T_1$. In addition, $X_0^T A X_1 - k_1 X_0^T A T_1 < 0$ and $X_1^T A X_0 + k_0 X_1^T A T_0 < 0$ ensure that proper weights for the two arcs of the biarc with joint Z can be solved for from (3). So, by Lemma 1.2, a biarc with proper weights can be constructed to interpolate D. □

For a general regular quadric S, we still do not have a geometric characterization as when the conditions in Theorem 3.4 are always satisfied. However, we will see that these conditions are always satisfied for generic data on a sphere.

3.2. *Biarcs for nonsingular data*

Definition 3.5. A biarc is *degenerate* if one of its arcs degenerates into a single point.

A biarc with control points X_0, Y_0, Z and Z, Y_1, X_1 for the two arcs is degenerate if and only if Z coincides with X_0 or X_1. The necessity is obvious. For the sufficiency suppose that $\langle Z\rangle = \langle X_1\rangle$ (the other case is similar). Then the control polygon ZY_1X_1 collapses into two coincidental line segments. First assume $k_1 \neq 0$. Then $Z^{\mathrm{T}}AX_1 = 0$ since $\langle Z\rangle = \langle X_1\rangle$, and $Y_1^{\mathrm{T}}AY_1 = k_1^2 \neq 0$. So $w = 0$ by (3), i.e., the arc controlled by ΔZY_1X_1 is a point. When $k_1 = 0$, by (5), $\langle Z\rangle = \langle Y_1\rangle = \langle X_1\rangle$, and again the arc becomes a point.

Theorem 3.6. *Let $D = \{X_0, X_1, T_0, T_1\}$ be data on a regular quadric S. Let k_0 and k_1, $k_0k_1 \neq 0$, be a solution of (13) such that $\langle Y_0\rangle \neq \langle Y_1\rangle$. Then the biarc interpolating D given by k_0 and k_1 is nondegenerate if and only if D is nonsingular.*

The proof of Theorem 3.6 is given by the following Lemma 3.7 and Lemma 3.8. The case where $\langle Y_0\rangle = \langle Y_1\rangle$ is excluded in Theorem 3.6 for the reason that in this case the argument leading to (13) is invalid. By Lemma 3.3 this case occurs only for singular data, and we will discuss it later on.

Lemma 3.7. *Let $D = \{X_0, X_1, T_0, T_1\}$ be singular data on a regular quadric S. Then Eq. (13) factors. And for any solution k_0 and k_1 of (13) such that $\langle Y_0\rangle \neq \langle Y_1\rangle$, the biarc interpolating D is degenerate.*

Proof. Let $D = \{X_0, X_1, T_0, T_1\}$ be singular. There are two cases to consider: (i) $X_0 + \rho T_0 = X_1 + \rho T_1$ for some finite $\rho \neq 0$; and (ii) $T_0 = T_1$.

(i) In this case $T_0 \neq T_1$. The left hand side of (13) becomes

$$\begin{aligned}
&X_0^{\mathrm{T}}A(X_0 + \rho T_0 - \rho T_1) + k_0(X_0 + \rho T_0 - \rho T_1)^{\mathrm{T}}AT_0 - k_1(X_0^{\mathrm{T}}AT_1) + \\
&\quad + k_0k_1(1 - T_0^{\mathrm{T}}AT_1) \\
&= -\rho(X_0^{\mathrm{T}}AT_1) + k_0\rho(1 - T_0^{\mathrm{T}}AT_1) - k_1(X_0^{\mathrm{T}}AT_1) + k_0k_1(1 - T_0^{\mathrm{T}}AT_1) \\
&= (k_1 + \rho)\left[k_0(1 - T_0^{\mathrm{T}}AT_1) - (X_0^{\mathrm{T}}AT_1)\right] \\
&= (k_1 + \rho)\left[k_0(1 - T_0^{\mathrm{T}}AT_1) - (X_1^{\mathrm{T}} + \rho T_1 - \rho T_0)^{\mathrm{T}}AT_1\right] \\
&= (k_1 + \rho)\left[k_0(1 - T_0^{\mathrm{T}}AT_1) - \rho(1 - T_0^{\mathrm{T}}AT_1)\right] \\
&= (1 - T_0^{\mathrm{T}}AT_1)(k_0 - \rho)(k_1 + \rho).
\end{aligned}$$

Since X_0 and X_1 are not on the same generating line,

$$\begin{aligned}
1 - T_0^{\mathrm{T}}AT_1 &= \frac{1}{2}(T_1 - T_0)^{\mathrm{T}}A(T_1 - T_0) = \frac{1}{2\rho^2}(X_0 - X_1)^{\mathrm{T}}A(X_0 - X_1) \\
&= -\frac{X_0^{\mathrm{T}}AX_1}{\rho^2} \neq 0.
\end{aligned}$$

So from (13) we have $k_0 = \rho$ or $k_1 = -\rho$. First we take $(k_0, k_1) = (\rho, k_1)$, $k_1 \neq -\rho$, as solutions of (13); here $k_1 \neq -\rho$ since $k_0 = \rho$ and $k_1 = -\rho$ would cause $Y_0 = Y_1$, a case that has been excluded. Therefore by (12),

$$\begin{aligned}
Z &= k_1(X_0 + k_0T_0) + k_0(X_1 - k_1T_1) = k_1(X_0 + \rho T_0 - \rho T_1) + \rho X_1 \\
&= (k_1 + \rho)X_1.
\end{aligned}$$

Thus $\langle Z\rangle = \langle X_1\rangle$ since $k_1 + \rho \neq 0$. Hence the resulting biarc is degenerate. When $(k_0, -\rho)$, with $k_0 \neq \rho$, are taken as solutions of (13), it can be shown similarly that $\langle Z\rangle = \langle X_0\rangle$.

(ii) Eq. (13) can be rewritten as

$$\frac{X_0^{\mathrm{T}}AX_1}{k_0k_1} + \frac{X_1^{\mathrm{T}}AT_0}{k_1} - \frac{X_0^{\mathrm{T}}AT_1}{k_0} + 1 - T_0^{\mathrm{T}}AT_1 = 0.$$

Since in this case $T_0 = T_1$, $1 - T_0^{\mathrm{T}}AT_1 = 1 - T_0^{\mathrm{T}}AT_0 = 0$, $X_0^{\mathrm{T}}AT_1 = X_0^{\mathrm{T}}AT_0 = 0$, and $X_1^{\mathrm{T}}AT_0 = X_1^{\mathrm{T}}AT_1 = 0$. The above equation is reduced to $X_0^{\mathrm{T}}AX_1/(k_0k_1) = 0$. This equation is satisfied by $k_0 = \pm\infty$ or $k_1 = \pm\infty$. First take a finite k_0 and $k_1 = \infty$ as solutions of (13); here k_0 is finite because when $k_0 = \pm\infty$ and $k_1 = \pm\infty$, we have $\langle Y_0\rangle = \langle Y_1\rangle = \langle T_0\rangle$, a case that has been excluded. Then

$$\begin{aligned}\langle Z\rangle &= \langle k_1(X_0 + k_0T_0) + k_0(X_1 - k_1T_1)\rangle = \langle k_1(X_0 + k_0T_0 - k_0T_1) + k_0X_1\rangle \\ &= \langle k_1X_0 + k_0X_1\rangle = \langle X_0\rangle.\end{aligned}$$

Hence the resulting biarc is degenerate. The case where $k_0 = \pm\infty$ and k_1 is finite can be proved similarly. □

Lemma 3.8. *Let $D = \{X_0, X_1, T_0, T_1\}$ be nonsingular data on a regular quadric S. Then the interpolating biarc given by any solution k_0 and k_1 of Eq. (13), $k_0k_1 \neq 0$, is nondegenerate.*

Proof. Suppose there is a degenerate biarc interpolating D, with $k_0k_1 \neq 0$. Without loss of generality, assume that $Z = \beta X_0$ for some $\beta \neq 0$. Then by (12),

$$\beta X_0 = k_1(X_0 + k_0T_0) + k_0(X_1 - k_1T_1).$$

Since X_0 and X_1 are in normalized homogeneous form and the last components of T_0 and T_1 are zero, $\beta = k_0 + k_1$. Thus

$$k_0X_0 - k_0k_1T_0 = k_0X_1 - k_0k_1T_1,$$

or, since $k_0 \neq 0$,

$$X_0 - k_1T_0 = X_1 - k_1T_1.$$

Since $k_1 \neq 0$, by definition, D is singular, a contradiction. Note that the above equation reduces to $T_0 = T_1$ if $k_1 = \infty$, again implying that D is singular data. □

3.3. Biarcs for singular data

About the existence of nondegenerate biarcs interpolating singular data, we have

Lemma 3.9. *Let B be a nondegenerate biarc interpolating singular data $D = \{X_0, X_1, T_0, T_1\}$ on a regular quadric. Let X_0Y_0Z and ZY_1X_1 be the control polygons of the two arcs of B, respectively. Then $Y_0 = Y_1$ and only one of the arcs has proper weight.*

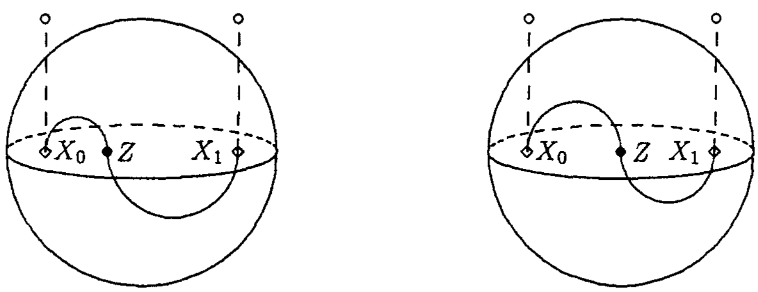

Fig. 7. Two biarcs interpolating singular data with $T_0 = T_1$.

Proof. $Y_0 = Y_1$ is implied by Lemma 3.7. As the joint Z is the tangent point of the line Y_0Z to the quadric S, Z is outside the degenerate line segment $\overline{Y_0Y_1}$ (since $Y_0 = Y_1$). So by Lemma 1.2, only one of the two arcs has proper weight. □

When $\langle Y_0 \rangle = \langle Y_1 \rangle$ for singular data $D = \{X_0, X_1, T_0, T_1\}$, the locus of Z is the intersection of S with the polar hyperplane $Y_0^{\mathrm{T}}AX = 0$ of Y_0 with respect to S, since Z is the tangent point to S of a straight line passing through Y_0. Let Z_D denote the locus of Z. For each point $Z \in Z_D$ but $Z \neq X_0$ and X_1, two rational Bézier curves on S with control polygons X_0Y_0Z and ZY_1X_1, respectively, can be constructed to join with G^1 continuity at Z. These two Bézier curves yield a nondegenerate biarc interpolating D if they are both continuous. Fig. 7 shows two biarcs on S^2 interpolating singular data with $T_0 = T_1$.

The degree of freedom of biarcs interpolating singular data, if they exist, is $d - 2$, which is the dimension of Z_D. However, the degree of freedom of biarcs interpolating nonsingular data, if they exist, is only one. Therefore, when $d > 3$, it is possible that there exist more interpolating biarcs for singular data than for nonsingular data. This is exactly the case on the sphere $S^{d-1} \subset \mathbb{E}^d$, $d > 3$.

3.4. Biarcs on a sphere

Theorem 3.10. *There exists a nondegenerate biarc with proper weights interpolating data $D = \{X_0, X_1, T_0, T_1\}$ on $S^{d-1} \subset \mathbb{E}^d$ if and only if D is nonsingular.*

Proof. The necessity is implied by Lemma 3.9. We now prove the sufficiency. By Lemma 3.8, since D is nonsingular, every biarc interpolating D with $k_0k_1 \neq 0$ is nondegenerate. It suffices to show that the conditions of Theorem 3.4 are always met on S^{d-1}.

Let the equation of S^{d-1} be $X^{\mathrm{T}}AX = 0$, with

$$A = \begin{bmatrix} I_d & 0 \\ 0 & -1 \end{bmatrix},$$

where I_d is the $d \times d$ identity matrix. Then $(Y_0^{\mathrm{T}} A Y_0)(X_0^{\mathrm{T}} A Z) < 0$ and $(Y_1^{\mathrm{T}} A Y_1)$ $(X_1^{\mathrm{T}} A Z) < 0$ always hold on S^{d-1}. Also, as $T_0^{\mathrm{T}} A T_0 = T_0^{\mathrm{T}} T_0 > 0$ and $T_1^{\mathrm{T}} A T_1 = T_1^{\mathrm{T}} T_1 > 0$, the assumption that $T_0^{\mathrm{T}} A T_0 = T_1^{\mathrm{T}} A T_1 = 1$ is justified. By Theorem 3.4, we just need to show that Eq. (13) has solutions k_0 and k_1 with $k_0, k_1 > 0$. Since X_0 and X_1 are in normalized homogeneous form and $X_0 \neq X_1$, we have

$$-2X_0^{\mathrm{T}} A X_1 = (X_0 - X_1)^{\mathrm{T}} A (X_0 - X_1) = (X_0 - X_1)^{\mathrm{T}} (X_0 - X_1) > 0.$$

So $X_0^{\mathrm{T}} A X_1 < 0$, i.e., the constant term Eq. (13) is negative. Since D is nonsingular, by definition, $T_0 \neq T_1$. Therefore

$$1 - T_0^{\mathrm{T}} T_1 = \frac{1}{2}(T_0 - T_1)^{\mathrm{T}}(T_0 - T_1) > 0. \tag{14}$$

That is, the coefficient of $k_0 k_1$ in (13) is positive. Hence (13) has positive solutions k_0 and k_1 because $k_0 = k_1 = 0$ makes the left hand side of (13) negative and sufficiently large positive values of k_0 and k_1 make it positive. □

Given nonsingular data D on S^{d-1}, setting $k_0 = k_1$ in (13), we have the equation

$$ak^2 + bk + c = 0, \tag{15}$$

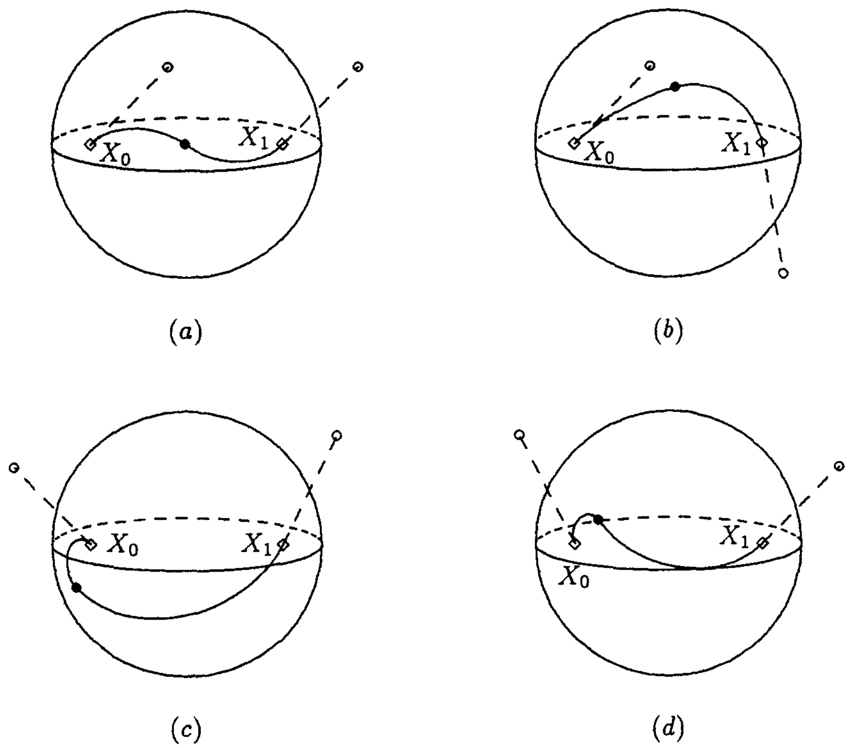

Fig. 8. Four different data configurations on sphere and their biarc interpolants. The joints are marked with •. The parameters k_0 and k_1 used are the positive root of Eq. (15).

where $a = 1 - T_0^T A T_1$, $b = X_1^T A T_0 - X_0^T A T_1$ and $c = X_0^T A X_1$. By the argument in the proof of Theorem 3.10, this equation has positive solution $k = [-b + (b^2 - 4ac)^{1/2}]/(2a)$ for nonsingular D on S^{d-1}. A particular positive solution of (13) is $k_0 = k_1 = k$. Fig. 8 shows the biarc interpolants on S^2 for several different data configurations, using the positive root of (15) as k_0 and k_1.

3.5. The locus of joint

When (13) does not factor into two linear factors, k_1 can be expressed in terms of k_0; then $Z(k_0, k_1)$ given by (12) is parametric curve of k_0.

Lemma 3.11. *For $D = \{X_0, X_1, T_0, T_1\}$ on a regular quadric S, Eq. (13) does not factor if and only if D is nonsingular.*

Proof. The necessity is given in the proof of Lemma 3.7. We will only outline the proof for the sufficiency part. Since a bilinear function $axy + bx + cy + d$, with $a \neq 0$, factors if and only if the discriminant $ad - bc = 0$, we just need to show that the discriminant of the left hand side of (13) does not vanish. Let h be the discriminant of (13), i.e.,

$$h = (X_0^T A X_1)(1 - T_0^T A T_1) + (X_0^T A T_1)(X_1^T A T_0).$$

Let $M = [X_0 T_0 X_1 T_1]^T A [X_0 T_0 X_1 T_1]$. Then it can be verified that

$$\det(M) = h\big[-(X_0^T A X_1)(1 + T_0^T A T_1) + (X_0^T A T_1)(X_1^T A T_0)\big].$$

When D is nonsingular, X_0, T_0, X_1 and T_1, being treated as four points in projective space, are either noncoplanar or coplanar. When they are noncoplanar, $\det(M) \neq 0$ (and hence $h \neq 0$), for otherwise, the 3-dimensional affine manifold spanned by the four points would be contained in the quadric S, contradicting that S is regular. When the four points are coplanar, noting that the last components of T_0 and T_1 are zero, and X_0 and X_1 are in normalized form, we have $X_0 + \rho_0 T_0 = X_1 + \rho_1 T_1$ for some ρ_0 and ρ_1. Then

$$(X_0 + \rho_0 T_0)^T A (X_0 + \rho_0 T_0) = (X_1 + \rho_1 T_1)^T A (X_1 + \rho_1 T_1),$$

or, after simplification, $\rho_0^2 = \rho_1^2$. It follows that $\rho_0 = -\rho_1$, since D is nonsingular. Letting $\rho = \rho_0 = -\rho_1$, we have $X_0 + \rho_0 T_0 = X_1 - \rho T_1$; obviously $\rho \neq 0$ since $X_0 \neq X_1$. Using this equality it can be verified directly that $h = 2X_0^T A X_1 \neq 0$. Thus $h \neq 0$ for any nonsingular data D. Hence Eq. (13) does not factor. □

Theorem 3.12. *For nonsingular data $D = \{X_0, X_1, T_0, T_1\}$ on a regular quadric S, the locus of Z given by (12) is a conic on S passing through X_0 and X_1.*

Proof. Since D is nonsingular, by Lemma 3.11, (13) is irreducible, so k_1 can be expressed in terms of k_0,

$$k_1 = \frac{X_0^T A X_1 + k_0 X_1^T A T_0}{X_0^T A T_1 + k_0 (T_0^T A T_1 - 1)}.$$

Substituting it in (12) and multiplying the denominator, we have

$$Z(k_0) = [X_0^{\mathrm{T}} A X_1 + k_0 X_1^{\mathrm{T}} A T_0]\big[X_0 + k_0(T_0 - T_1)\big] + k_0\big[X_0^{\mathrm{T}} A T_1 + k_0(T_0^{\mathrm{T}} A T_1 - 1)\big]X_1. \tag{16}$$

So the locus of Z is a rational quadratic curve on S. Since $Z = (X_0^{\mathrm{T}} A X_1)X_0$ when $k_0 = 0$, the locus passes through X_0. By a similar argument, it also passes through X_1. □

The biarc interpolant has conic precision in the sense that when data $D = \{X_0, X_1, T_0, T_1\}$ is extracted from a conic arc C on a quadric S, any point on C is the joint of a biarc that reproduces C. Since, by Theorem 3.12, the locus of the joint Z is a conic, this locus must be the underlying conic of the arc C.

3.6. Interpolating a sequence of points

In Section 2 we see that when there exists a rational quadratic spline curve interpolating a point sequence $\{X_i\}_{i=1}^{n}$ on a quadric, it is determined by a global parameter $Y_1 \in L_1$, which has $(d-2)$ degrees of freedom. This property is quite undesirable because any local perturbation of the data or change of Y_1 has global influence on the curve. Now we consider using the biarc interpolant to obtain a locally controllable conic spline curve interpolating $\{X_i\}_{i=1}^{n}$.

Algorithm 3.1. Given $\{X_i\}_{i=1}^{n}$ on the same component of a regular quadric $S \subset \mathbb{E}^d$, $d \geqslant 3$. Assume that no two consecutive points X_i and X_{i+1} are on the same generating line of S, $i = 1, 2, \ldots, n-1$.

(1) Determine a tangent vector T_i at X_i as the unit tangent vector to the conic $\widetilde{C}_i$ on S interpolating the three points X_{i-1}, X_i and X_{i+1}, $i = 1, 2, \ldots, n$. The direction of T_i conforms with the direction along which a point moves on the conic $\widetilde{C}_i$ from X_{i-1} through X_i to X_{i+1}. $X_0 = X_3$ and $X_{n+1} = X_{n-2}$ are assumed to provide the end tangent directions T_1 and T_n.

(2) Use a biarc with proper weights to interpolate $D_i = \{X_i, X_{i+1}, T_i, T_{i+1}\}$, $i = 1, \ldots, n-1$. □

Fig. 9 shows an interpolating spline curve given by Algorithm 3.1 for the same data points as in Fig. 4.

A property of the above algorithm is that a conic section is locally reproduced; that is, if X_{i-1}, X_i, X_{i+1} and X_{i+2} are on any conic C on S, C is reproduced by the algorithm between X_i and X_{i+1}, $i = 1, \ldots, n-1$.

The above algorithm works correctly for any point sequence on a quadric S which is affinely equivalent to S^{d-1}. When S is a general quadric, there exists a biarc with proper weights interpolating X_i and X_{i+1} if the conditions of Theorem 3.4 are satisfied.

In the second step of Algorithm 3.1, if there exist biarcs interpolating D_i, we have to choose one of them according to some criterion. A satisfactory choice entails a study of the influence of the parameters k_0 and k_1 on the shape of the resulting biarc. On a general quadric this problem is still under investigation.

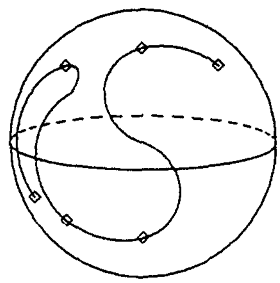

Fig. 9. The same data points in Fig. 4 is interpolated using the biarcs given by Algorithm 3.1.

4. Concluding remarks

Two interpolation problems on a quadric are studied. In the first problem we consider using a rational quadratic spline curve to interpolate a point sequence on a regular quadric $S \subset \mathbb{E}^d$, $d \geqslant 3$. Given a point sequence $\{X_i\}_{i=1}^n$, $n \geqslant 3$, on the same real component of $S: X^T AX = 0$, it is shown that a necessary condition on the existence of a rational quadratic spline curve on S interpolating $\{X_i\}_{i=1}^n$ is $(X_1^T AX_2)(X_i^T AX_{i+1}) > 0$, $i = 1, 2, \ldots, n-1$, or geometrically, all the line segments $\overline{X_i X_{i+1}}$, $i = 1, 2, \ldots, n-1$, are on the same side of S. This condition is sufficient and is always satisfied when S is affinely equivalent to a sphere.

For the second problem, we use a biarc to interpolate distinct points X_0, X_1 and tangents T_0, T_1 specified at X_0 and X_1, respectively. It is shown that for generic data the biarc interpolant has one degree of freedom. A necessary and sufficient condition on the existence of the biarc with proper weights is given. This condition is satisfied by generic data on the sphere $S^{d-1} \subset \mathbb{E}^d$, $d \geqslant 3$.

Several open problems still remain. We have shown that not every point sequence on a general quadric admits interpolation by the rational quadratic spline curve. If this kind of data occurs, other methods have to be used for interpolation.

For the biarc interpolation problem we have given a sufficient and necessary algebraic condition on th existence of a biarc interpolant with proper weights (Theorem 3.4). Yet we do not know how restrictive this condition is on a general quadric in terms of geometric characterization.

Appendix

To prove Lemma 2.5, we need the following lemma.

Lemma A.1. *Let A be a real $n \times n$ nonsingular symmetric matrix. Let A have p positive and r negative eigenvalues, $p + r = n$. Let B be an $n \times (n-1)$ matrix of rank $n-1$. Then the symmetric matrix $B^T AB$ has at least $p-1$ positive and at least $r-1$ negative eigenvalues.*

Proof. Since the rank of B is $n-1$, we can add a new column b to it such that $D = [B, b]$ is nonsingular. Then $B^{\mathrm{T}}AB$ is the leading $(n-1)\times(n-1)$ principal submatrix of $D^{\mathrm{T}}AD$. By the Sylvester law of inertia (Golub and Van Loan, 1989, pp. 416–417), the number of positive eigenvalues and the number of negative eigenvalues of $D^{\mathrm{T}}AD$ are the same as those of A. Since the eigenvalues of $B^{\mathrm{T}}AB$ separate those of $D^{\mathrm{T}}AD$ (Wilkinson, 1965, pp. 103–104), we conclude that $B^{\mathrm{T}}AB$ has at least $p-1$ positive eigenvalues and at least $r-1$ negative eigenvalues. □

Proof of Lemma 2.5. To simplify notation, in this proof we shall use X_0 and X_1 to replace X_i and X_{i+1}. First we need an affine classification of real quadrics in $\mathbb{E}^d$ (Xu, 1965, pp. 471–474). It is straightforward to show that any real regular quadric in $\mathbb{E}^d$ is affinely equivalent to one of the following forms:

(1) $X^{\mathrm{T}}AX = 0$, where $A = \mathrm{diag}[1, \sigma_2, \ldots, \sigma_d, -1]$, $\sigma_i = \pm 1,\ i = 2, \ldots, d$; or
(2) $X^{\mathrm{T}}AX = 0$, where

$$A = \mathrm{diag}\left[\sigma_1, \ldots, \sigma_{d-1}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\right], \quad \sigma_i = \pm 1,\ i = 1, \ldots, d-1.$$

Let A have p positive and r negative eigenvalues. Since A is indefinite for $X^{\mathrm{T}}AX = 0$ to be a real surface, $p \geqslant 1$ and $r \geqslant 1$.

We just have to show that the lemma holds for surfaces in these two canonical forms. First consider the class of quadrics $X^{\mathrm{T}}AX = 0$ with $p = 1$ or $r = 1$. These quadrics must be in one of the following three cases:

(1) $X^{\mathrm{T}}AX = 0$ with $A = \mathrm{diag}[I_d, -1]$;
(2) $X^{\mathrm{T}}AX = 0$ with $A = \mathrm{diag}[1, -I_d]$, which gives the same quadric as by $A = \mathrm{diag}[-1, I_d]$;
(3) $X^{\mathrm{T}}AX = 0$ with

$$A = \mathrm{diag}\left[I_{d-1}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\right].$$

For these three cases, $p = d$ and $r = 1$. Define a point X_0 to be inside S: $X^{\mathrm{T}}AX = 0$ if $X_0^{\mathrm{T}}AX_0 < 0$. Let S be any one of the above three surfaces. Then, given any two distinct real points $X_0 = [x_{0,1}, \ldots, x_{0,d}, 1]^{\mathrm{T}}$, $X_1 = [x_{1,1}, \ldots, x_{1,d}, 1]^{\mathrm{T}}$ on the same component of S, it will be shown that the line segment $\overline{X_0X_1}$ is inside S.

(1) The case $A = \mathrm{diag}[I_d, -1]$:

$$(X_0 + X_1)^{\mathrm{T}}A(X_0 + X_1) = 2X_0^{\mathrm{T}}AX_1 = -(X_0 - X_1)^{\mathrm{T}}A(X_0 - X_1) < 0.$$

(2) The case $A = \mathrm{diag}[-1, I_d]$: Since $x_1 = 0$ is the separating hyperplane of S, S has two components. Since X_0, X_1 are on the same component of S, we have $x_{0,1}x_{1,1} > 0$. Then

$$\begin{aligned} x_{0,1}x_{1,1}(X_0 + X_1)^{\mathrm{T}}A(X_0 + X_1) &= 2x_{0,1}x_{1,1}X_0^{\mathrm{T}}AX_1 \\ &= -(x_{1,1}X_0 - x_{0,1}X_1)^{\mathrm{T}}A(x_{1,1}X_0 - x_{0,1}X_1) < 0. \end{aligned}$$

Thus $(X_0 + X_1)^{\mathrm{T}}A(X_0 + X_1) < 0$.

(3) The case

$$A = \operatorname{diag}\left[I_{d-1}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\right].$$

Since X_0, X_1 are real points on S, $x_{0,d} > 0$ and $x_{1,d} > 0$. Then

$$\begin{aligned}
&-(x_{0,d}+1)(x_{1,d}+1)(X_0+X_1)^{\mathrm T}A(X_0+X_1)\\
&= -2(x_{0,d}+1)(x_{1,d}+1)X_0^{\mathrm T}AX_1\\
&= \big[(x_{1,d}+1)X_0-(x_{0,d}+1)X_1\big]^{\mathrm T}A\big[(x_{1,d}+1)X_0-(x_{0,d}+1)X_1\big]\\
&= \sum_{i=1}^{d-1}\big[(x_{1,d}+1)x_{0,i}-(x_{0,d}+1)x_{1,i}\big]^2 -\\
&\quad -2\big[(x_{1,d}+1)x_{0,d}-(x_{0,d}+1)x_{1,d}\big]\big[(x_{1,d}+1)-(x_{0,d}+1)\big]\\
&= \sum_{i=1}^{d-1}\big[(x_{1,d}+1)x_{0,i}-(x_{0,d}+1)x_{1,i}\big]^2 + 2(x_{1,d}-x_{0,d}) > 0,
\end{aligned}$$

since X_0, X_1 are distinct points. Thus $(X_0+X_1)^{\mathrm T}A(X_0+X_1) < 0$ since $x_{0,d}+1 > 0$ and $x_{1,d}+1 > 0$. Hence for the three quadrics the segment $\overline{X_0X_1}$ is inside the surface.

Let $L = Q_0 \cap Q_1$, where Q_0 and Q_1 are the tangent hyperplanes of S at X_0 and X_1, given by $X_0^{\mathrm T}AX = 0$ and $X_1^{\mathrm T}AX = 0$, respectively. Let Y_i, $i = 1, \ldots, d-1$, be $d-1$ affinely independent points in L. Similar to the Gram–Schmidt orthogonalization process (Golub and Van Loan, 1989, p. 218), the Y_i can be constructed so that $Y_i^{\mathrm T}AY_j = 0$ for $i \neq j$. Let $Z = \lambda X_0 + \mu X_1$ be a variable point on the straight line X_0X_1. Then $Z^{\mathrm T}AY_i = 0$ for $i = 1, \ldots, d-1$. Let $B = [Y_1, \ldots, Y_{d-1}, Z]$. Then $B^{\mathrm T}AB = \operatorname{diag}[Y_1^{\mathrm T}AY_1, \ldots, Y_{d-1}^{\mathrm T}AY_{d-1}, Z^{\mathrm T}AZ]$. Since $X_0^{\mathrm T}AX_1 \neq 0$, $Z^{\mathrm T}AX_0 = \mu X_1^{\mathrm T}AX_0 \neq 0$ or $Z^{\mathrm T}AX_1 = \lambda X_0^{\mathrm T}AX_1 \neq 0$; therefore $Z \notin L$. So B has rank d. By Lemma A.1, $B^{\mathrm T}AB$ has at least $d-1$ positive eigenvalues since A has d positive eigenvalues. Since Z changes sign at X_0 or X_1, it can be chosen so that $Z^{\mathrm T}AZ < 0$; therefore the $Y_i^{\mathrm T}AY_i > 0$. Thus $Y^{\mathrm T}AY > 0$ for any $Y \in L$. Hence when S is any of the three quadrics, for any $Y \in L$, the line segment $\overline{X_0X_1}$ and Y are on opposite sides of S.

Now consider the remaining case, i.e., the quadrics $X^{\mathrm T}AX = 0$ with $p \geqslant 2$ and $r \geqslant 2$. Let $B = [Y_1, \ldots, Y_{d-1}, Z]$ be the same as constructed above. For the same reason, B has rank d and $B^{\mathrm T}AB = \operatorname{diag}[Y_1^{\mathrm T}AY_1, \ldots, Y_{d-1}^{\mathrm T}AY_{d-1}, Z^{\mathrm T}AZ]$. In this case, by Lemma A.1, $B^{\mathrm T}AB$ has at least one positive eigenvalue and one negative eigenvalue; so it is indefinite. Therefore the $Y_i^{\mathrm T}AY_i$ do not have the same sign; for otherwise, choosing $Z^{\mathrm T}AZ$ to have the same sign as the $Y_i^{\mathrm T}AY_i$, $B^{\mathrm T}AB$ would become positive or negative definite, a contradiction. Since the $Y_i^{\mathrm T}AY_i$ have different signs, there exists $Y \in L$ such that $\overline{X_0X_1}$ and Y are on opposite sides of S. Then the lemma follows from Theorem 2.3. □

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